The exponential nature of a bouncing ping-pong ball

I learned an amazing story about exponential grow in my mathematics class. There is a fascinated geometric phenomena described in the legend of a king and a chessboard.

> *"According to legend, a courtier presented the Persian king with a beautiful, hand made chessboard. The king asked what he would like in return for his gift and the courtier surprised the king by asking for one grain of rice on the first square, two grains on the second, four grains on the third, etc. The king readily agreed and asked for the rice to be brought. All went well at first, but the requirement for 2^{n-1} grains on the nth square demanded over a million grains on the 21st square, more than a million-million (or trillion) on the 41st and there simply was not enough rice in the whole world for the final squares."*

http://www.dr-mikes-math-games-for-kids.com/rice-and-chessboard.html

Exponential Growth and Decay

The chessboard story is an example of exponential growth. This occurs when the growth rate of a mathematical function is proportional to the function's current value. This is also known as geometric growth. Similarly, there is exponential decay when the rate of decrease of a mathematical function is proportional to the function's current value.
http://en.wikipedia.org/wiki/Exponential_growth/

Consider a barrel holding 100 gallons of water. A leak occurs and after 5 minutes only 50 gallons remain. After an additional 5 minutes the 50 gallons is reduced to 25 gallons. And, after another 5 minutes, the 25 gallons is reduced to 12.5 gallons. This continues with the same mathematical pattern, that is, half the remaining water leaves the barrel in the same amount of time no matter how much or how little water is in the barrel.

Exponential growth and decay are fundamental properties in many aspects of nature. In biology this occurs with microorganisms in a culture, with the growth of a virus, sometimes in human population growth. In the field of economics exponential change occurs in finance; and it is found in computer technology. In physics the charge and discharge of capacitors follow exponential functions, and this relationship is well known in nuclear reactions and radioactive decay, and in heat transfer. Many other areas of natural phenomena can be described as exponential. That is why I love this topic!

Exponential Decay in This Investigation

In this investigation I will consider the **exponential decay of a bouncing ball**. The exponential mathematical model of the bouncing ball will be confirmed, although I did not know this before the experiment. The **bouncing half-life** (the number of bounces before the rebound height reaches one-half a previous height) will also be determined.

To clearly focus this approach to the motion of a bouncing ball I will consider only vertical bounces. The drop height will also limit the investigation; I use only one initial drop height and take only one set of data. Too high of an initial drop introduces noticeable spin and too low an initial drop does not provide enough data.

1

Bouncing Balls

Under normal bouncing conditions (a reasonable height and a uniform surface) balls rebound to a height that is less than the drop height because kinetic energy is lost on the rebound. The ball never experiences an elastic collision. Some heat and sound are always produced. We also know that there is a characteristic unique to a given ball for the amount of height lost between the drop height and the rebound height. This might be called the 'bounciness' or the percentage of rebound. It is related to the coefficient of restitution.

These percentages of rebound express an exponential property for a bouncing ball. A golf ball, for instance, will bounce back to 36% of the drop height. See the information found on the web site: http://www.exploratorium.edu/baseball/bouncing_balls.html. Each consecutive bounce will reach 36% of the previous height, and so on.

Preliminary Work

Ball Selection. A variety of balls were found in the physics lab, and each was dropped and the bouncing was observed, both in terms of keeping along the vertical and making a clear sound or well-defined sound upon impact. I decided that a Ping-Pong ball was the best ball to use.

Drop Height. The Ping-Pong ball dropped from about 60 cm made clear impact sounds and produced sufficient repetitive bouncing. This value was arbitrary and does not play a role any of my calculations, only for collecting sufficient data.

Rebound Height. This will be calculated based on the time interval between two consecutive bounces.

Time Measurement. The sound of the bounce impacts is recorded using a microphone, data logging interface and a computer. I used the *Vernier* (see http://www.vernier.com/) Lab Pro interface and the Logger Pro software with my Mac computer. This is illustrated in below in Graph 3.

Percentage of Rebound Test. A preliminary test was made by dropping the Ping-Pong ball from a variety of heights and measuring the rebound height using the *Vernier* sonic motion detector. If the percentage of rebound is consistent then the exponential model may well be appropriate. The results are shown below in Graph 1.

GRAPH 1: Rebound Height against Drop Height

2
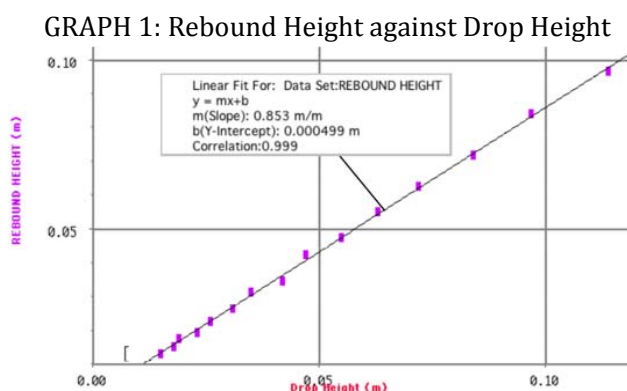
The data reveals a linear function with a gradient of 0.853. This is the same as saying that the ball experiences 85.3% efficiency on its bounce for a variety of drops heights. The systematic shift in all the data is simply an artifact of the measuring technique and does not affect the gradient value. It is safe, therefore, to assume that this rebound percentage is a characteristic of the Ping-Pong ball and hence my exponential model is appropriate.

Experimental Design

The purpose of this investigation is to confirm the exponential nature of a bouncing ball and thereby calculate the half-life of the bouncing ball. I will limit my study to one type of ball, and I will limit the study to just one initial drop height and the resulting bouncing heights. Because the ball's rebound height decreases with time this is the same as having a number of different drop heights in sequential order; that is to say, for example, that rebound number 4 is the drop height of bounce number 5, and so on.

Using the microphone, the *LabPro* data-logging interface and computer system, the sound of bounces was recorded. Heights were calculated by the spreadsheet. The ball bounce events were simply counted; the first bounce as $n = 1$, the second was $n = 2$, etc. When a rebound height became one-half the height of some previous height then it will have bounced a number of times n (not necessarily a whole number, perhaps $n = 3.33$) indicating its half-life. With radioactive decay, the half-life is a function of time (seconds, minutes, years) but with a bouncing ball the event of a bounce is the measure.

Experimental Variables

In order to determine the consecutive heights I recorded the sound of the bouncing ball. Upon impact, the ball made an impact sound. This is the same as saying the air pressure is constant while there is no sound, but changes or is offset from this constant when there is sound. Sound is longitudinal wave, a compression and rarefaction of air pressure.

The **independent variable** is the bounce number. The life of a bouncing ball is measured as the 1st, 2nd, 3rd, etc. bounce number. This is a counting number, a pure number with no units and no significant uncertainties. However, identifying the moment of impact is limited to the precision of the sampling rate, so there is a minimum uncertainty, about 1 ms in this study.

The **dependent** variable is the rebound height, H, the maximum height reached between bounces. This was calculated in units of metres (m). To measure H, the time ΔT between consecutive bounces was determined (from a graph of air pressure offset as a function of time) and H calculated from $H = \frac{1}{2}gt^2$ where $t = \frac{1}{2}\Delta T$. The $\frac{1}{2}$ comes from the fact that ΔT is the time up to the rebound height plus the time down from the rebound height. It is far more accurate to measure this time interval, and then calculate the height, than it would be to try to measure the rebound height of a moving bouncing ball. There is no significant uncertainty in the calculated height as it is based on a very precise timing

3

mechanism with the computer and interface. The offset of air pressure is recorded at a rate of 1000 measurements per second.

The **controlled** variables included using the same surface and the same ball. This was obvious as I made only one data set. Time limited repeated measures. Nonetheless, I made a number of trials and selected the cleanest data set, and one where the bouncing was more or less always along the vertical. This is good scientific practice thus demonstrating the nature of science at work. If the ball moved off the vertical while bouncing, then the data was rejected. A controlled variable was that the bouncing stays more or less along the vertical.

Four Experimental Assumptions

(1) The evidence for exponential decay has already been discussed and it is a reasonable model for the bouncing ball. This was confirmed in Graph 1. I then select one drop height and let the ball bounce again and again and then see if this motion is indeed exponential. This is **the first mathematical model assumption** I made.

(2) The value of free-fall gravity will be assumed in the calculations. If this is in error it will be a systematic error, a constant, and hence will make no difference to the results.

(3) It is true by definition that the distance up equals the distance down. However, if the ball spins and moves off a purely vertical path then the data trail was rejected. Again, in this experiment a number of trials were done but only one set of data was used to determine the results.

(4) If the distance up equals the distance down then assuming uniform acceleration we can say that the time up equals the time down. This assumes that air friction plays no role in the motion of the ball. From a practical point of this, air friction can be ignored, but technically the time up is not equal to the time down if we account for air friction. On the upward journey the weight is directed downward and the air resistance retards the motion; air friction is directed downwards. The net force causing the ball to decelerate is then $F_{\text{weight}} + F_{\text{air}}$. When the ball falls from the maximum height to the ground the weight is again directed downwards but the air resistance, which retards the ball's motion, is directed upwards. Hence the force causing the ball to accelerate is $F_{\text{weight}} - F_{\text{air}}$. This means that the time going up is less than the time coming down when air friction is accounted for. In my experiment the balls maximum speed is rather small, and the height distance is small too, and I assume air resistance is negligible. Moreover, a test with the sonic motion detector (*Vernier*) revealed that any asymmetry between the up and down times is equal the period of the sampling frequency. That is, no difference is detected other than the precision of 1 ms and any noise or random effects, artifacts of the measuring process. My **second mathematical model assumption** then is the basic equation of uniform accelerated motion, relating height h, gravity g and time t, as $h = \frac{1}{2} g t^2$.

4

Gathering Data

The time ΔT is determined by recording the impact sound of a bouncing ball. Time intervals are read off a graph of sound pressure against time.

FIGURE 2: Sample of Raw Data

This shows a sample of the raw data of the air pressure (in arbitrary units), the offset pressure when there is sound, and time measurements (in seconds). The microphone was connected to the Vernier *Lab Pro* interface and then this was connected to the computer. The *Vernier's LoggerPro* 3.4.1 software automatically sensed the microphone and displayed graph axes of sound level against time. The sampling rate was set at 1000 Hz. http://www.vernier.com/.

|	Time (s)	Pres (arbitrary)	Offset (arbitrary)
1	0.000	2.6837600	0.0002516
2	0.001	2.6837600	0.0002516
3	0.002	2.6837600	0.0002516
4	0.003	2.6825400	-0.0009684
5	0.004	2.6837600	0.0002516
6	0.005	2.6837600	0.0002516
7	0.006	2.6837600	0.0002516
8	0.007	2.6837600	0.0002516
9	0.008	2.6837600	0.0002516
10	0.009	2.6825400	-0.0009684
11	0.010	2.6825400	-0.0009684
12	0.011	2.6837600	0.0002516
13	0.012	2.6837600	0.0002516
14	0.013	2.6849800	0.0014716

The sampling rate was set at 1000 Hz and I took 6 seconds of data. Only a spreadsheet could handle all this data. See the last datum at number 6000 in the figure below.

FIGURE 3: Last Few Data Values

5998	5.997	2.6825400	-0.0009684
5999	5.998	2.6825400	-0.0009684
6000	5.999	2.6837600	0.0002516

Close-Up View of Graphed Data

Graph 2: Close Up of Air Pressure Offset against Time
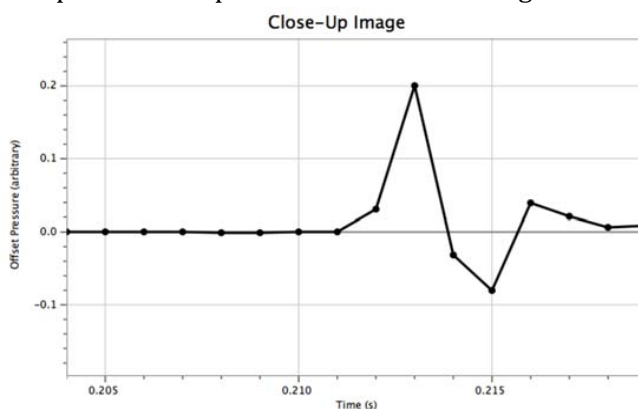
This is a close-up view of the next graph. By clicking on a peak (maximum or minimum intensity offset) the time datum is given. Although there are several data points before the peak (where the ball first hits the table) I used the first maximum or minimum peak (greatest offset intensity) in all the time measurements.

Graphing Data

The following graph (Graph 3) includes all 6000 data points.

5

GRAPH 3:
Graph of Offset Sound Pressure (arbitrary units) against Time (s)
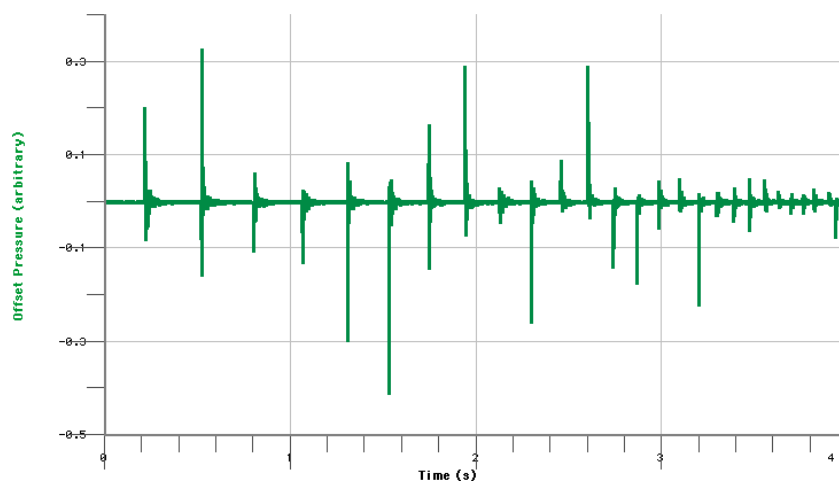
The sharp peaks, both positive and negative, indicate a pulse of sound; they represent the sound made at impact. I decided to use the largest datum value as the impact time. Even if there is a systematic error here, using the same reference point for all measurements of time makes this assumption reliable. The computer calculates the consecutive times for the first peaks of each bounce and uses this to calculate the rebound height H and the natural logarithm of the height H. A value of $g = 9.81\,\mathrm{m\,s^{-2}}$ is used but as this is a constant through out the experiment, it could have been normalized, i.e. $g = 1$.

Processing Data

The processing of data was done in the spreadsheet automatically once I programmed the calculations. The *Logger Pro* graphing program nicely determined consecutive values as well as doing the various calculations.

FIGURE 4: Spreadsheet Processed Data

	N	Total Time (s)	Delta Time (s)	Height (m)	ln (Height)	H(n+1) / H(n)
1	0	0.213				
2	1	0.518	0.305	0.114	-2.173	
3	2	0.799	0.281	0.097	-2.337	0.851
4	3	1.061	0.262	0.084	-2.477	0.866
5	4	1.303	0.242	0.072	-2.636	0.857
6	5	1.529	0.226	0.063	-2.773	0.875
7	6	1.741	0.212	0.055	-2.900	0.873
8	7	1.938	0.197	0.047	-3.047	0.855
9	8	2.124	0.186	0.042	-3.162	0.894
10	9	2.292	0.168	0.035	-3.366	0.833
11	10	2.452	0.160	0.031	-3.463	0.886
12	11	2.599	0.147	0.026	-3.633	0.837
13	12	2.735	0.136	0.023	-3.788	0.885
14	13	2.861	0.126	0.019	-3.941	0.826
15	14	2.981	0.120	0.018	-4.039	
16	15	3.092	0.111	0.015	-4.195	
17	16	3.195	0.103	0.013	-4.344	
18	17	3.294	0.099	0.012	-4.423	
19	18	3.384	0.090	0.010	-4.614	

6

The following shows of how the spreadsheet made the calculations. I use just one set of values to illustrate this.

Delta Time Calculation:

$$\text{Delta Time} = \text{Total Time}_{N+1} - \text{Total Time}_N$$

$$\text{Delta Time} \rightarrow \quad \Delta T_{\text{N to N+1}} = T_{N+1} - T_N$$

For example, the interval from N = 3 to N = 4 is as follows.

$$\Delta T_{3 \rightarrow 4} = 1.303\text{s} - 1.061\text{s} = 0.242\text{s}$$

Rebound Height Calculation:

$$\text{Rebound Height} = H = \frac{1}{2}gt^2 = \frac{1}{2}g\left(\frac{\Delta T}{2}\right)^2$$

$$H = \frac{g\Delta T^2}{8} = \frac{\left(9.81\,\text{m s}^{-2}\right)\Delta T^2}{8} = 1.22625 \times \Delta T^2$$

For example, H for the interval $N = 3 \rightarrow N = 4$ is as follows.

$$H = 1.22625\left(\Delta T_{3 \rightarrow 4}\right)^2 = 1.22625\left(0.242\text{s}\right)^2 = 0.071814\,\text{m} \approx 0.072\,\text{m}$$
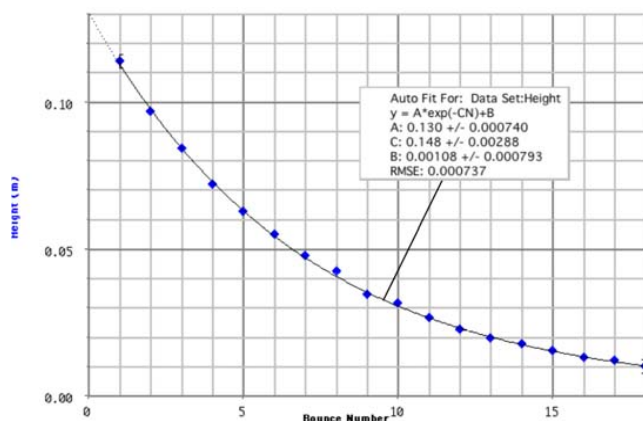
Natural Logarithm of Height Calculation:

$$\ln h_{3 \rightarrow 4} = \ln(0.071814) \approx -2.477$$

Analysis of Data

The calculated heights are now graphed against bounce number. You can see the gradient decreases as the bounce number increases. This could be exponential.

GRAPH 4: Height (m) against Bounce Number

From Graph 3 we can see, for example, that H reduces from 0.10 m to 0.05 m in about 5 bounces (4.9) and then from 0.05 to 0.025 in a further 5 bounces, thus indicating an exponential decay. That is, changes from H to $\frac{1}{2}H$ have equal bounce intervals, the half-life of the bouncing ball.

7

Assume the mathematical model where $H = H_0 e^{-\lambda n}$ and λ is the decay constant and n is the bounce number so that a plot of the natural logarithm of H against n should give a straight line graph the gradient of which is $= \lambda$. This graph (Graph 4) is plotted below.

Logarithmic Graph

GRAPH 4: Natural Logarithm of Height against Bounce Number
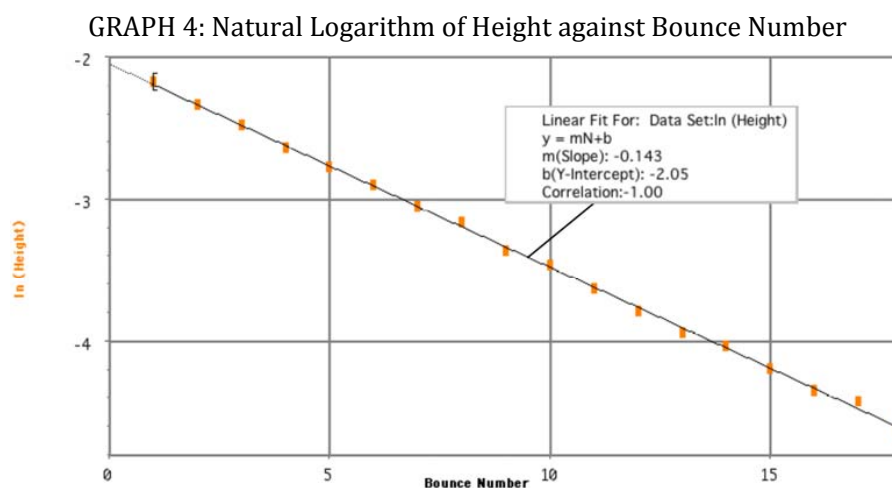
The best-fit line is clearly linear indicating an exponential relationship. Note that both logarithms and counting numbers have no units; they are pure numbers.

Half-Life Analysis

The computer calculated the gradient m of the graph where $m = \lambda = -0.143$. The half-life is calculated from $t_{\frac{1}{2}} = \dfrac{\ln 0.5}{\lambda} = \dfrac{-0.693}{-0.143} = 4.846$ bounces ≈ 4.8 bounces.

There is no reason to expect the half-life of height to be a whole number of bounces. In fact, as the ball continues to bounce the time between bounces decreases. This decay in time is not exponential. A graph of the natural logarithm of height against time reveals this but is not included here.

Uncertainty in The Exponential Model

Instead of propagating the errors based on precision and random estimates, I decided to graph the characteristic that I assumed in my exponential model, namely the ratio of rebound against bounce number, and then find a value of uncertainty from the scatter of these values (one unique value per bounce event, see Graph 5 below). The average of this can easily be determined and an uncertainty for the average can be found from the range of ratio values.

8

GRAPH 5: Rebound Ratio against Bounce Number
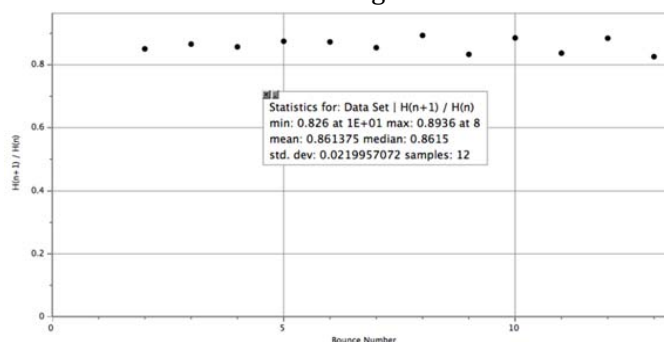
The *LoggerPro* software statistical analysis function tells us the minimum and the maximum values of this rebound ratio. It also tells us the average or mean value, which is 0.861375 or 86.1%.

The uncertainty in this can be found using the range of date as follows, where *y* is the mean value of the ratio.

$$\pm\Delta y = \pm\frac{y_{max} - y_{min}}{2} = \pm\frac{0.8936 - 0.8260}{2} = \pm0.0338$$

$$y \pm \Delta y = 0.861375 \pm 0.0338 \approx 0.86 \pm 0.03$$

As a percentage, we find: $\frac{0.0338}{0.861375} \times 100\% = 3.92395\% \approx 4\%$

The percentage of rebound and its uncertainty are thus (87±4)%. However, if we take a statistical approach and use standard deviation, we get a ratio and its uncertainty as 0.86 ± 0.02 or only 2.3%. When rounded off this is about 2%, which is half the uncertainty when calculated by the range method. I can assume that this limit is due to random errors as well as less than ideal exponential motion (air resistance, slight spin, etc.).

In both sets of uncertainty calculations I assume that under perfect conditions the exponential model applies. My assumptions has been confirmed to a high degree but not proved. Recall the 38 arc-seconds difference between Newton's and Einstein's theories of gravity when it came to the perihelion precession of the planet Mercury. Small details can make big differences in theory. And now, for my **Conclusions** and **Evaluative** comments.

9

The Exponential Model. The experimental results make a beautiful fit of the exponential model. The consecutive decreases in rebound heights follow the model, and this was estimated as being within about 2% uncertainty, a most acceptable conclusion.

The Half-Life of a Bouncing Ball. Following the exponential trend, the half-life of a bouncing ball (the number of bounces for the rebound to become half an initial value) was determined to be about 4.84 bounces. No uncertainties were determined for this.

Obtaining Results. The measurements would not have been possible without the data-logging and a computer as well as the data-processing using a spreadsheet. Using a meter stick and eyeing the bouncing ball would have been impossible.

Limitations. One limitation in this experiment is that there was insufficient time to take more data in the form of repeated trials from the same initial drop height. No doubt the data would vary slightly due to possible ball spin produced by the release method, slightly different impact conditions (perhaps in a repeated trail the ball is every so slightly warmer), and so on. I doubt if the results would be different.

One advantage of repeated trials would be to establish a degree of confidence, an uncertainty in the results by error propagation method. More data would have helped to test the validity and or limitations of the exponential model. I could have also tested this model's validity for balls of different material dropped on different surfaces.

An important improvement would be to record more precise intervals of sound, perhaps changing the sample rate from 1000 Hz to 2000 Hz. Also, assuming slight spin caused the slight variation in the data, some sort of mechanical release mechanism might help improve the already high degree of precision.

My preliminary investigations showed me that much higher drop heights (as well as very low drop heights) do not produce quality data, so my conclusions in this investigation are limited to the scope and range that I covered.

Further studies, or an extension to this, would be to determine the half-life of different types of balls. Now that I have thoroughly analyzed the Ping-Pong ball, I a happy to leave my study there.

Overall, I was amazed that I could clearly demonstrate the exponential nature of a bouncing ball, just one more fascinating example of this mathematical function found in the natural world.

10